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# FOREIGN TECHNOLOGY DIVISION

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by

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| Block | Italic | Transliteration | Block | Italic | Transliteration |
|---|---|---|---|---|---|
| А а | *А а* | A, a | Р р | *Р р* | R, r |
| Б б | *Б б* | B, b | С с | *С с* | S, s |
| В в | *В в* | V, v | Т т | *Т т* | T, t |
| Г г | *Г г* | G, g | У у | *У у* | U, u |
| Д д | *Д д* | D, d | Ф ф | *Ф ф* | F, f |
| Е е | *Е е* | Ye, ye; E, e* | Х х | *Х х* | Kh, kh |
| Ж ж | *Ж ж* | Zh, zh | Ц ц | *Ц ц* | Ts, ts |
| З з | *З з* | Z, z | Ч ч | *Ч ч* | Ch, ch |
| И и | *И и* | I, i | Ш ш | *Ш ш* | Sh, sh |
| Й й | *Й й* | Y, y | Щ щ | *Щ щ* | Shch, shch |
| К к | *К к* | K, k | Ъ ъ | *Ъ ъ* | " |
| Л л | *Л л* | L, l | Ы ы | *Ы ы* | Y, y |
| М м | *М м* | M, m | Ь ь | *Ь ь* | ' |
| Н н | *Н н* | N, n | Э э | *Э э* | E, e |
| О о | *О о* | O, o | Ю ю | *Ю ю* | Yu, yu |
| П п | *П п* | P, p | Я я | *Я я* | Ya, ya |

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

# PLANE STRESSED STATE AND BENDING OF NONHOMOGENEOUS TRANSVERSALLY ISOTROPIC PLATE

S. G. Lekhnitskiy

(Leningrad)

Generalization of the theory of elastic equilibrium of a thick plate [1, 2] is given for nonhomogeneous transversally isotropic plates of two types: a) multilayer and b) having elastic moduli, continuously changing in thickness. There are considered plates, being in plane stressed state and bent by loads distributed along the edges arbitrarily, and also uniformly along one of the plane surfaces. The constructed theories of plane stressed state and bending do not use hypothesis of straight normals and others (besides those usually taken in the linear theory of elasticity) and permit determining stresses and displacements, strictly satisfying all equations of the theory of elasticity, conditions on plane surfaces and on contact surfaces of layers. On lateral surface (on the edge) generally there are satisfied averaged or integral conditions, as in the theory of thin plates.

1. General information. Let us examine a multilayer plate, comprised of an odd number of 2n + 1 elastic transversally isotropic layers, located symmetrically relative to the middle. Let us take the following limitations:

1) Each two layers, symmetric relative to the middle, have identical thickness and identical elastic properties, which planes

of isotropy in every layer are parallel to the middle plane of the entire plate.

2) Layers are soldered or glued, so that their slip along contact surfaces and separation from each other are eliminated.

3) All the layers behave as elastic, and experience small deformations, connected with stresses by generalized Hooke law.

4) Poisson's ratio for planes of isotropy (characterizing reduction in the plane of isotropy with extension in the same plane) for all layers are identical.

The last limitation is very essential, and without it, it is not possible to construct the simple theory, which is discussed below.

Let us take the middle plane for plane xy and direct axis z downwards (Fig. 1). Let us number the layers in the following way: let us consider the middle zero (0), let us assign the underlayer numbers 1, 2, ..., n, and overlayer – numbers -1, -2, ..., -n.

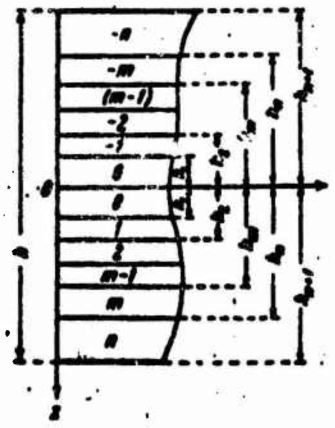

Fig. 1.

In accordance with this, stresses and elastic constants will be designated by usual symbols with the addition of numbers of layers on top, and displacements of any point of the layer number

m — through $u_m$, $v_m$, $w_m$. Let us introduce more designations: h — total thickness of plate, $h_m$ — distance from middle plane to contact surface of layers m and m - 1; let us assume that $\delta_m = h_{m+1} - h_m$ — thickness of layers with numbers m and -m. We have:

$$
\begin{gathered}
E^{-m} = E^m, \quad E_1^{-m} = E_1^m \quad \text{(Young's moduli)} \\
G_1^{-m} = G_1^m, \quad G^{-m} = G^m = \frac{E^m}{2(1+\nu)} \quad \text{(shear moduli)} \\
\nu^1 = \nu^2 = \ldots = \nu^{-1} = \nu^{-2} = \ldots = \nu^m = \nu, \quad \nu_1^{-m} = \nu_1^m, \quad \nu_2^{-m} = \nu_2^m \\
E_1^m \nu_2^m = E^m \nu_1^m \quad (\nu \text{ — Poisson's ratios}).
\end{gathered} \tag{1.1}
$$

Equations of generalized Hooke law for layer number m (m = 0, ±1, ±2, ..., ±n) will be written in the following way [3]:

$$
\begin{aligned}
\varepsilon_x^m &= \frac{1}{E^m}(\sigma_x^m - \nu\sigma_y^m) - \frac{\nu_1^m}{E_1^m}\sigma_z^m, & \gamma_{yz}^m &= \frac{1}{G_1^m}\tau_{yz}^m \\
\varepsilon_y^m &= \frac{1}{E^m}(\sigma_y^m - \nu\sigma_x^m) - \frac{\nu_1^m}{E_1^m}\sigma_z^m, & \gamma_{xz}^m &= \frac{1}{G_1^m}\tau_{xz}^m \\
\varepsilon_z^m &= -\frac{\nu_1^m}{E^m}(\sigma_x^m + \sigma_y^m) + \frac{1}{E_1^m}\sigma_z^m, & \gamma_{xy}^m &= \frac{1}{G^m}\tau_{xy}^m
\end{aligned} \tag{1.2}
$$

Basic system of equations of the theory of elasticity for transversally isotropic layer number m will be obtained by adding to (1.2) the equations of equilibrium ($X_m$, $Y_m$, $Z_m$ — projections of volume of forces)

$$
\begin{aligned}
\partial_1\sigma_x^m + \partial_2\tau_{xy}^m + \partial_3\tau_{xz}^m + X_m &= 0 \\
\partial_1\tau_{xy}^m + \partial_2\sigma_y^m + \partial_3\tau_{yz}^m + Y_m &= 0 \\
\partial_1\tau_{xz}^m + \partial_2\tau_{yz}^m + \partial_3\sigma_z^m + Z_m &= 0
\end{aligned} \tag{1.3}
$$

Here and further

$$
\begin{gathered}
\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z} \\
D^2 = \partial_1^2 + \partial_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\end{gathered} \tag{1.4}
$$

On plane surfaces z = ±h/2 stress components should equal projections of assigned forces (per unit of area). On contact surfaces there should be fulfilled conditions

$$\tau_{xz}^{m} = \tau_{xz}^{m-1}, \quad \tau_{yz}^{m} = \tau_{yz}^{m-1}, \quad \sigma_z^{m} = \sigma_z^{m-1}$$
$$u_m = u_{m-1}, \quad v_m = v_{m-1}, \quad w_m = w_{m-1} \tag{1.5}$$

Let us also examine single-layer transversally isotropic plate with thickness h with planes of isotropy parallel to the middle xy, for which $\nu$ = const, and the remaining elastic moduli and Poisson's ratio $E$, $E_1$, $G_1$, $\nu_1$, $\nu_2$ are continuous and even functions of variable z, so that $E(-z) = E(z)$, etc. Let us assume that this plate also experiences small elastic deformations and follows generalized Hooke law. Basic system of equations of the theory of elasticity for it will be obtained from (1.2) and (1.3), having rejected there indices - numbers of layers.

For each of the two plates - layered and with variable moduli - let us examine three cases of equilibrium: 1) plane stressed state, induced by forces distributed along the edge, 2) bending by load distributed along the edge, and 3) bending by normal load, distributed evenly over the entire surface $z = -h/2$. Plane stressed state and bending are examined separately for multilayer plates and plates with variable elastic moduli. Below are listed only final formulas for stresses and displacements, satisfying equations (1.2) and (1.3), conditions on surfaces $z = \pm h/2$ and on contact surfaces (in multilayer plate) and equations for function of stresses and sag of the middle plane. The fact that these functions satisfy the needed equations and conditions can be easily checked by simple substitution. The below mentioned expressions give the possibility of satisfying averaged or integral conditions on the edge.

Thus, as in the case of homogeneous transversally isotropic plate, the first two of the named problems lead to determination of biharmonic function in the region of plate satisfying boundary conditions, and the third problem - to determination of the function satisfying equation of a thin plate, bent by evenly distributed load.

In the given strict formulation the problem for anisotropic nonhomogeneous plate is examined, apparently, for the first time. In general, according to the theory of multilayer plates and shells (especially – sandwich) there are known many works, in which different assumptions are used (see [4], and also [5] and [6], where there are large lists of literature). This question, just as the question about bending, will be examined separately for multilayer plates and for plates with variable elastic moduli.

2. Plane stressed state. *a) Multilayer plate.* Let us assume there is a multilayer plate with the above-mentioned properties, of arbitrary shape (Fig. 1) and with arbitrary number of layers 2n + 1 being in equilibrium under the action of forces, distributed along the lateral surface symmetrically relative to the middle plane and parallel to it. More precisely, it is assumed that forces affecting any element hds of the lateral surface with base ds and height equal to the thickness of plate, are brought to principal vector, lying in the middle plane. Volume forces are not examined.

In homogeneous isotropic or transversally isotropic plate, being in the same conditions [3], stress components on areas parallel to the middle plane are equal to zero, and the others are quadratic functions of z.

In multilayer symmetric plate we also have

$$\tau_{xz}^{m} = \tau_{yz}^{m} = \sigma_z^{m} = 0 \qquad (2.1)$$

for every layer. For average stresses and displacements[1] along the thickness the same equations are obtained as for isotropic plate with Poisson's ratio $\nu$ and Young's modulus $E^{\times}$, equal to

$$E^{\times} = \frac{1}{h}\left(E^{0}\delta_0 + 2\sum_{k=1}^{n} E^{k}\delta_k\right), \qquad (2.2)$$

[1]Subsequently, average values will be designated by superscript "slanting cross."

namely

$$\sigma_x^\times = \partial_2^2 F^\times,\quad \sigma_y^\times = \partial_1^2 F^\times,\quad \tau_{xy}^\times = -\partial_1\partial_2 F^\times \tag{2.3}$$

$$E^\times \partial_1 u^\times = (\partial_2^2 - \nu\partial_1^2) F^\times,\quad E^\times \partial_2 v^\times = (\partial_1^2 - \nu\partial_2^2) F^\times$$

$$E^\times \cdot (\partial_1 v^\times + \partial_2 u^\times) = -2(1+\nu)\,\partial_1\partial_2 F^\times \tag{2.4}$$

Function of stresses $F^\times$ is biharmonic, i.e., satisfies equation

$$D^4 F^\times = 0 \tag{2.5}$$

Let us introduce designations:

$$c = \frac{1}{E^\times h}\left\{\frac{E^0 \nu_z^0}{3} h_0^3 + \sum_{m=1}^{n} E^m\left[\frac{\nu_z^m}{3}(h^3_{m+1} - h_m^3) + \right.\right.$$
$$\left.\left. + h_m \sum_{k=1}^{m} (\nu_z^{k-1} - \nu_z^k) h_k (h_{m+1} + h_m - h_k)\right]\right\} \tag{2.6}$$

$$Z_0(z) = c - \frac{\nu_z^0}{2} z^2,\quad Z_{\pm m}(z) = c + \sum_{k=1}^{m} (\nu_z^{k-1} - \nu_z^k) h_k \left(\frac{h_k}{2} \mp z\right) - \frac{\nu_z^m}{2} z^2$$
$$(m = 1, 2, \ldots, n)$$

$$\Phi_{\pm m} = \frac{E^m}{E^\times}\left[F^\times + \frac{1}{1+\nu} Z_{\pm m}(z)\cdot D^2 F^\times\right] \quad (m = 0, 1, 2, \ldots, n) \tag{2.7}$$

Then formulas for stresses and displacements in every layer will be written in the following way:

$$\sigma_x^{\pm m} = \partial_2^2 \Phi_{\pm m},\quad u_{\pm m} = u^\times - \frac{1}{E^\times} Z_{\pm m}(z)\cdot\partial_1 D^2 F^\times$$
$$\sigma_y^{\pm m} = \partial_1^2 \Phi_{\pm m},\quad v_{\pm m} = v^\times - \frac{1}{E^\times} Z_{\pm m}(z)\cdot\partial_2 D^2 F^\times \quad (m = 0, 1, \ldots, n) \tag{2.8}$$
$$\tau_{xy}^{\pm m} = -\partial_1\partial_2 \Phi_{\pm m},\quad w_{\pm m} = \frac{1}{E^\times} Z'_{\pm m}(z)\cdot D^2 F^\times$$

*b) Plate with variable elastic moduli.* For a single-layer plate, for which $\nu$ = const, and remaining Poisson's ratios and elastic moduli are continuous and even functions of z, we obtain also

$$\tau_{xz} = \tau_{zy} = \sigma_z = 0 \tag{2.9}$$

Average stresses and displacements will be found from equations (2.3) and (2.4), where given modulus

$$E^{\times} = \frac{2}{h}\int_0^{h/2} E\,dz \tag{2.10}$$

Expressions for stresses and displacements have the form

$$\sigma_x = \partial_2^2\Phi,\quad \sigma_y = \partial_1^2\Phi,\quad \tau_{xy} = -\partial_1\partial_2\Phi \quad \left(\Phi = \frac{Z(z)}{E^{\times}}\left[F^{\times} + \frac{Z'(z)}{1+\nu}\cdot D^2F^{\times}\right]\right) \tag{2.11}$$

$$u = u^{\times} - \frac{Z(z)}{E^{\times}}\cdot\partial_1 D^2F^{\times},\quad v = v^{\times} - \frac{Z(z)}{E^{\times}}\cdot\partial_2 D^2F^{\times},\quad w = -\frac{Z'(z)}{E^{\times}}\cdot D^2F^{\times} \tag{2.12}$$

Here

$$Z(z) = \frac{1}{E^{\times}h}\int_{-h/2}^{h/2} E\varkappa\,dz - \varkappa(z),\qquad \varkappa(z) = \int_0^z\left(\int_0^z \nu_z\,dz\right)dz \tag{2.13}$$

3. Bending by a load distributed along the edge. *a) Multilayer plate.* If a multilayer plate (Fig. 1) is bent by forces and moments, distributed along the edge, then stresses and displacements at any point can be expressed through biharmonic function W – sag of middle plane.

Let us indicate all the necessary formulas, preliminarily introducing the following shortened designations for constants and functions of z.

$$\begin{gathered}
a_m = \frac{1}{2G_1^m(1-\nu^2)}\left[G_1^m\sum_{k=1}^{m}(\nu_3^{k-1}-\nu_3^k)h_k^2 - \sum_{k=m+1}^{n}(E^k-E^{k-1})h_k^3 + E^n h^3_{n+1}\right] \\
(m = 1, 2, \ldots, n-1) \\
a_0 = \frac{1}{2G_1^0(1-\nu^2)}\left[-\sum_{k=1}^{n}(E^k-E^{k-1})h_k^3 + E^n h^3_{n+1}\right] \qquad d_0 = 0 \\
a_n = \frac{1}{2G_1^n(1-\nu^2)}\left[G_1^n\sum_{k=1}^{n}(\nu_3^{k-1}-\nu_3^k)h_k^2 + E^n h^3_{n+1}\right] \\
c_m = \frac{1}{2(1-\nu)}\left(\frac{2G^m}{G_1^m}-\nu_3^m\right),\quad d_m = \sum_{k=1}^{m}(c_{k-1}-c_k)h_k + \frac{1}{3}\sum_{k=1}^{m}(a_k-a_{k-1})h_k^3 \\
(m = 1, \ldots, n)
\end{gathered} \tag{3.1}$$

$$\zeta_{km} = zW + \left(\pm d_m + c_m z - \frac{a_m}{3}z^3\right)D^2W \tag{3.2}$$

$$F_m(z) = \frac{E^m}{3}z^3 + \left[\sum_{k=m+1}^{n}(E^k-E^{k-1})h_k^3 - E^n h^3_{n+1}\right]z \qquad (m=0, 1, \ldots, n-1) \tag{3.3}$$

$$F_m(z) = \frac{E^m}{2}(z^2 - 3h^2_{m+1}), \quad B = \frac{2}{3(1-\nu^2)}\left[E^0 h_1^3 + \sum_{k=1}^{n} E^k (h^3_{k+1} - h_k^3)\right] \tag{3.4}$$

$$g = \frac{2}{15(1-\nu^2)B}\left\{E^0 (5c_0 - a_0 h_1^2)\, h_1^3 + \sum_{k=1}^{n} E^k [7.5 d_k (h^4_{k+1} - h_k^4) + \right.$$
$$\left. + 5c_k (h^3_{k+1} - h_k^3) - a_k (h^5_{k+1} - h_k^5)]\right\}. \tag{3.5}$$

Expressions for displacements in any layers have the form

$$\begin{gathered} u_{\pm m} = -\partial_1 \zeta_{\pm m}, \quad v_{\pm m} = -\partial_2 \zeta_{\pm m} \qquad (m = 0, 1, \ldots, n) \\ w_{\pm m} = W + \frac{1}{2(1-\nu)}\left[\nu_m z^2 + \sum_{k=1}^{m} (\nu_k^{k-1} - \nu_k^k) h_k^2\right] D^2 W \quad (m = 1, \ldots, n) \\ w_0 = W + \frac{\nu^0}{2(1-\nu)} z^2 D^2 W \end{gathered} \tag{3.6}$$

For stresses we obtain formulas

$$\begin{gathered} \sigma_x^{\pm m} = -\frac{E^m}{1-\nu^2}(\partial_1^2 + \nu\partial_2^2)\zeta_{\pm m}, \quad \sigma_z^{\pm m} = 0 \qquad (m = 0, 1, \ldots, n) \\ \sigma_y^{\pm m} = -\frac{E^m}{1-\nu^2}(\partial_2^2 + \nu\partial_1^2)\zeta_{\pm m}, \quad \tau_{xz}^{\pm m} = \frac{F_m'(z)}{2(1-\nu^2)} \cdot \partial_1 D^2 W \\ \tau_{xy}^{\pm m} = -\frac{E^m}{1-\nu^2}(1-\nu)\partial_1\partial_2\zeta_{\pm m}, \quad \tau_{yz}^{\pm m} = \frac{F_m'(z)}{2(1-\nu^2)} \cdot \partial_2 D^2 W \end{gathered} \tag{3.7}$$

Stresses $\sigma_x$, $\sigma_y$, $\tau_{xy}$ will be reduced to moments, and $\tau_{xz}$, $\tau_{yz}$ – to shear forces. After integration with respect to thickness we obtain formulas for moments and shear forces, pertaining to unit of length:

$$\begin{gathered} M_x = -B(\partial_1^2 + \nu\partial_2^2) w', \quad M_y = -B(\partial_2^2 + \nu\partial_1^2) w' \\ H_{xy} = -B(1-\nu)\partial_1\partial_2 w' \\ N_x = -B\partial_1 D^2 W \quad N_y = -B\partial_2 D^2 W \quad (w' = W + gD^2 W) \end{gathered} \tag{3.8}$$

Coefficient B represents cylindrical rigidity of the plate.

Expression B coincides with what is obtained in the theory of thin multilayer plates on the basis of hypothesis of straight normals (see [7]).

*b) Plate with variable elastic moduli.* For such a plate instead of formulas (3.6), (3.7) we obtain

$$u = -\theta_1 \zeta, \quad v = -\theta_2 \zeta, \quad w = W + \frac{1}{1-\nu}\int_0^z \nu_z z\,dz \cdot D^2 W \tag{3.9}$$

$$\sigma_x = -\frac{E(z)}{1-\nu^2}\left(\theta_1^2 + \nu\theta_2^2\right)\zeta, \quad \sigma_z = 0$$

$$\begin{gathered}\sigma_y = -\frac{E(z)}{1-\nu^2}\left(\theta_2^2 + \nu\theta_1^2\right)\zeta, \quad \tau_{xz} = \frac{1}{1-\nu^2}\int_{h/2}^{z} Ez\,dz \cdot \theta_1 D^2 W \\ \tau_{xy} = -\frac{E(z)}{1-\nu^2}(1-\nu)\,\theta_1\theta_2\zeta, \quad \tau_{yz} = \frac{1}{1-\nu^2}\int_{h/2}^{z} Ez\,dz \cdot \theta_2 D^2 W\end{gathered} \tag{3.10}$$

$$\zeta = zW + \frac{1}{1-\nu}\int_0^z \left[\int_0^z \nu_z z\,dz - \frac{1}{(1+\nu)G_1}\int_{h/2}^{z} Ez\,dz\right] dz \cdot D^2 W = zW + f(z)\,D^2 W \tag{3.11}$$

Bending and twisting moments are determined by formulas (3.8), where expressions for B and g have the form

$$B = \frac{2}{1-\nu^2}\int_0^{h/2} Ez^2\,dz, \quad g = \frac{1}{(1-\nu^2)B}\int_{-h/2}^{h/2} Ez f(z)\,dz \tag{3.12}$$

(Expression for B is analogous to theory of plates, see [7], p. 272.)

From all the above-mentioned formulas the elastic moduli, constant at any point of the plate, are obtained by known formulas for homogeneous transversally isotropic plate. The problem on bending of multilayer isotropic plate is briefly examined (by a somewhat different method) in book [8].

4. Remarks in connection with bending of plate by distributed load. If the load is distributed not only along the edges, but also along planes $z = \pm h/2$ of nonhomogeneous plate, then the problem is considerably complicated (just as in the case of homogeneous plate); it can be easily solved by elementary means only with simple laws of distribution of load. Let us indicate briefly the basic results for an evenly loaded plate.

*a) Multilayer plate.* Let us assume that the plate is somehow attached along the edge and is loaded by normal forces, distributed

evenly along the surface $z = -h/2$ (Fig. 2). Let us take into account its intrinsic weight.

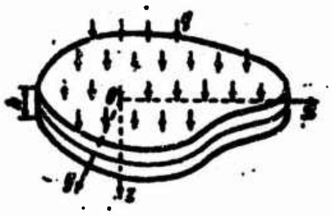

Fig. 2.

Let us assume that $q$ – intensity of load, $\gamma_m$ = const – specific gravity of layer number m, with this $\gamma_{-m} = \gamma_m$. Let us introduce more designations:

$$Q = \gamma_0\delta_0 + 2\sum_{k=1}^{n}\gamma_k\delta_k, \quad \lambda = \frac{1}{B^{\times}k}\left(\nu_2^0\delta_0 + 2\sum_{k=1}^{n}\nu_2^k\delta_k\right), \quad \mu_m = \frac{1-\nu-2\nu_1^m\nu_2^m}{E_1^m} \tag{4.1}$$

$$\sigma_z^{\pm m} = -\frac{1}{2(1-\nu^2)B}(q+Q)\left\{F_m(z) \pm \frac{2}{3}\sum_{k=1}^{m}(E^k - E^{k-1})h_k^3\right\} - \gamma_m z \pm \sum_{k=1}^{m}(\gamma_k - \gamma_{k-1})h_k \quad (m = 1, \ldots, n) \tag{4.2}$$

$$\sigma_z^0 = -\frac{1}{2(1-\nu^2)B}(q+Q)F_0(z) - \gamma_0 z; \quad \varphi_{\pm m}(z) = \int_0^z \sigma_z^{\pm m}dz \tag{4.3}$$

In this case sag is no longer a biharmonic function, but satisfies equation

$$BD^4W = q + Q \tag{4.4}$$

Displacements of points of layer number $\pm m$ will be obtained by adding to expressions (2.8) $u_{\pm m}$, $v_{\pm m}$ respectively

$$u'_{\pm m} = 0.5q\lambda x, \quad v'_{\pm m} = 0.5q\lambda y \tag{4.5}$$

and to $w_{\pm m}$ – function

$$w'_{\pm m} = \frac{q}{2(1-\nu)}\left\{-(\mu_m + 2\nu_2^m\lambda)z \pm \sum_{k=1}^{m}[\mu_k - \mu_{k-1} + 2\lambda(\nu_2^k - \nu_2^{k-1})]h_k\right\} + \frac{1}{1-\nu}\left\{\mu_m\varphi_{\pm m}(z) + \sum_{k=1}^{m}[\mu_{k-1}\varphi_{k-1}(h_k) - \mu_k\varphi_k(h_k)]\right\} \quad (m \neq 0) \tag{4.6}$$

or accordingly

$$w'_0 = -\frac{q}{2B(1-\nu)}(p_0 + 3m_0\nu_0)z + \frac{\nu_0}{1-\nu}\varphi_0(z) \qquad (4.5a)$$

Formulas for $\tau_{xz}$, $\tau_{yz}$, $\tau_{xy}$ will not be changed, but to $\sigma_{x^{\pm m}}$, $\sigma_{y^{\pm m}}$ it is necessary to add expression

$$\sigma'_{\pm m} = -\frac{q}{2}\frac{\lambda E^m - \nu_0^m}{1-\nu} + \frac{\nu_0^m}{1-\nu}\sigma_0^{\pm m} \qquad (4.7)$$

which, naturally, will be reflected on the moments. Furthermore, here stress $\sigma_z$ will not be equal to zero, but will be determined by formula

$$\sigma_z^{\pm m} = -\frac{q}{2} + \sigma_z^{\pm m} \qquad (4.8)$$

*b) Plate with variable elastic moduli.* At constant specific gravity $\gamma$ the sag of middle plane W satisfies equation (4.4), where Q = $\gamma h$, and B is calculated by formula (3.12). With q = const

$$\sigma_z = -\frac{q+Q}{(1-\nu)B}\int_{h/2}^{z}\left(\int_{h/2}^{z} Ez\,dz_1\right)dz + \gamma\left(\frac{h}{2} - z\right) \qquad (4.9)$$

Displacements will be determined by formulas (3.9) with addition of terms

$$u' = \lambda q x, \quad v' = \lambda q y \quad \left(\lambda = \int_{-h/2}^{h/2}\nu_z\sigma_z\,dz \Big/ 2q\int_0^{h/2} E\,dz\right) \qquad (4.10)$$

$$w' = \frac{q+\gamma h}{(1-\nu)B}\int_0^z \nu_0 f(z)\,dz + \int_0^z\left(\frac{1-2\nu_1\nu_2}{E_1}\sigma_z - \frac{2q\lambda}{1-\nu}\nu_0\right)dz$$

(expression f(z) – see (3.11)).

Stresses $\tau_{xz}$, $\tau_{yz}$, $\tau_{xy}$ will be determined by the same formulas as with q = 0, and to the expressions for $\sigma_x$ and $\sigma_y$ it is necessary to add

$$\sigma'_x = \sigma'_y = \frac{q\Delta E + \nu\sigma_z}{1-\nu} \tag{4.11}$$

Problem abound bending of isotropic plate with modulus E, changing in thickness, was examined in [9].

While not citing results of calculations for nonhomogeneous plates, let us note only that with different relationships between elastic constants of layers the greatest stress and especially the greatest sag $W_{max}$, obtained by strict theory, can differ quite considerably from those found with the help of theory of thin plates. As shown by formulas (3.8), moments depend on constant g (see (3.5) and (3.12)), which in different variants of nonhomogeneous plates can change over a wide range.

By solving, for example, the problem of bending of a supported plate, we require that along the edge the sag and bending moment be equal to zero and we thereby introduce constant g into expressions for displacements and tensions, which can be strongly reflected on their value.

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ABSTRACT (UNCL, O) ABSTRACT OF REPORT. A multilayer plate consisting of an odd number of transversely isotropic layers arranged symetric ally with respect to the median plane is considered under the following assumptions: 1) the pairs of layers which are symmetrical with respect to the median plane have equal thicknesses and the same elastic properties and the planes of isotropy in each layer are parallel to the median plane of the entire plate; 2) the layers are attached to each other along the contacting surfaces in such a way that slip or complete separation is impossible; 3) the general ized Hook's law holds; 4) the Poisson coefficients for the planes of isotropy of all the layers are the same. The second problem considered is that of a single layer transversely isotropic plate with the planes of isotropy parallel to the median plane in which the Poisson coefficient remains constant, while the remaining elastic moduli and Poisson coefficients are continuous and even functions of depth in the plate. Again, it is assumed that the generalized Hook's law holds. Two cases of equilibrium are then considered for both the multi layer and single layer plates, namely, the plane stressed state and bending under the action of stresses distributed arbitrarily over the lateral surface and bending under the action of a normal load over one of the plane faces. The analysis is a generalization of published solutions for the elastic equilibrium of a thick plate to the restricted non homogeneous case defined. (Two figures are included in the parent document, which is available on microfiche.)